UDC 531.36

ON A STATEMENT OF A STABILITY PROBLEM ON A FINITE INTERVAL

PMM Vol. 43, No. 1, 1979, pp. 172-175
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(Received June 5, 1978)

Sufficient stability conditions for a motion on a finite time interval in Lebedev's setting [1 — 3] are investigated. It is shown that for stability in the setting given it is sufficient that the matrix $P(t)$ of the linear part of the equations of perturbed motion have at the initial instant t_0 at least one eigenvalue with a negative real part, independently of the other eigenvalues of matrix $P(t_0)$, of the matrix $P(t) - P(t_0)$, and of the vector of constantly-acting perturbations, under the single condition that the components of this vector be bounded in modulus by sufficiently small positive quantities.

Lebedev [1 — 3] proposed the following statement of the stability problem for motion on a finite interval. Let the perturbed motion of some dynamic system be described by a differential equation system in the matrix form

$$dx/dt = P(t)x + h(t, x) + g(t, x) \tag{1}$$

where x is a column matrix of deviations (perturbations) $x_1, x_2, \ldots, x_n$; $P(t)$, $h(t, x)$ and $g(t, x)$ are matrices of dimensions $n \times n$, $n \times 1$ and $n \times 1$ respectively, being continuous and real functions of their arguments in the domain

$$T_1 \leqslant t \leqslant T_2, \ |x_i| \leqslant a_i \ (T_2 \neq T_1, \ a_i \neq 0)$$

In addition, $h(t, x) = 0(x)$, while the vector-valued function $g(t, x)$ with components $g_1(t, x), \ldots, g_n(t, x)$ characterizing the unknown perturbing forces, does not necessarily vanish for all x_i equalling zero. It is assumed that to the unperturbed motion whose stability is being investigated there corresponds a zero solution $x \equiv 0$ of the equation

$$dx/dt = P(t)x + h(t, x) \tag{2}$$

obtained from Eq. (1) by dropping function $g(t, x)$.

D e f i n i t i o n [1 — 3]. The unperturbed motion determined by the trivial solution of Eq. (2) is said to be stable on the finite interval $[t_0, t_0 + \tau]$ under constantly-acting perturbations if for every positive number c, however small, there exist a positive number $\eta(c)$ and a cycle $V(t, x) = c^2$ such that on this interval the diameter $D(t)$ of the domain

$$V(t, x) \leqslant c^2 \tag{3}$$

does not exceed the initial diameter $D(t_0)$ and every solution $x(t)$ of Eq. (1) with an initial condition $x_0 = x(t_0)$ satisfying the condition $V(t_0, x_0) \leqslant c^2$ satisfies inequality (3) under any perturbing forces $g_i(t, x)$ satisfying the condition

$$|g_i(t, x)| \leqslant \eta(c)$$

in domain (3) for $t_0 \leqslant t \leqslant t_0 + \tau$

As applied to the system of differential equations of perturbed motion in the absence of perturbing forces, the stability definition in the setting given was presented by Kamenkov and Lebedev in [4]. In [2] Lebedev indicated sufficient conditions for the existence of a stability interval in the linear approximation: for this it is sufficient that all roots of the characteristic equation $\det [P(t_0) - \lambda E] = 0$ have negative real parts. However, the stability conditions in the setting being examined are considerably broader. As Rudakov [5] showed, if even one element on the main diagonal of matrix $P(t_0)$ is negative, then a nonzero interval $[t_0, t_0 + \tau]$ exists on which the unperturbed motion is stable independently of the other element of matrix $P(t_0)$ of the matrix $P(t) - P(t_0)$, and of the vector $g(t, x)$. Let us show that for stability on a finite interval, in Lebedev's setting, it is sufficient that there exist at least one eigenvalue of matrix $P(t_0)$ with a negative real part, independently of the other eigenvalues of this matrix, as well as independently of the matrix $P(t) - P(t_0)$ and the vector $g(t, x)$.

Theorem. If at least one of the eigenvalues of matrix $P(t_0)$ has a negative real part, then a nonzero interval $[t_0, t_0 + \tau]$ exists on which the unperturbed motion (the trivial solution of Eq. (2)) is stable (in the sense of Lebedev's definition) independently of the other eigenvalues of matrix $P(t_0)$, of the matrix $P(t) - P(t_0)$, and of the vector $g(t, x)$.

Proof. Let λ_1 be an eigenvalue of matrix $P(t_0) = P_0$ and let $\mathrm{Re}\lambda_1 = -\alpha$ ($\alpha > 0$). We denote the eigenvector of matrix P_0, corresponding to this eigenvalue, by K_1. The eigenvector K_1 generates a one-dimensional invariant subspace R_1 of the n-dimensional Euclidean space R^n over the complex number field. In the $(n-1)$-dimensional invariant subspace R_2 of space R^n, orthogonal to vector K_1, we select some system of mutually orthogonal vectors $K_2, K_3, \ldots, K_n$, i.e., such that (the asterisk denotes the Hermitian conjugate of the matrix)

$$K_i^* K_j = 0 \quad (i, j = 2, \ldots, n)$$

We take it that all the vectors K_j $(j = 1, 2, \ldots, n)$ have been normed, so that $\| K_j \| = \sqrt{K_j^* K_j} = 1$. Under these conditions $K_1, K_2, \ldots, K_n$ form a orthonormalized system and $K = (K_1, K_2, \ldots, K_n)$ is a unitary matrix.

We define the domain of admissible states by means of the positive-definite Hermitian form

$$V(t, x) = (K_0^{-1}(t)x, \; K_0^{-1}(t)x)$$
$$K_0(t) = K\Omega(t), \quad \Omega(t) = \mathrm{diag}\,(1, \varepsilon(t) E_{n-1})$$

where E_{n-1} is the unit matrix of order $n-1$ and $\varepsilon(t)$ is some differentiable function bounded from below by a positive constant. The matrix of Hermitian form $V(t, x)$ is

$$A(t) = (K_0^{-1})^* K_0^{-1} = K\Omega^{-2}(t) K^{-1} \tag{4}$$

Let $\gamma_{\min}(t)$ be the smallest eigenvalue of matrix $A(t)$ at each instant t. Then, as is well known, the diameter of the domain

$$V(t, x) = x^* A(t) x \leqslant c^2 \tag{5}$$

is determined by the formula

$$D(t) = D(t_0) \sqrt{\gamma_{\min}(t_0) / \gamma_{\min}(t)} \tag{6}$$

From (4) and (6) we see that if $\varepsilon(t) < 1$ on the interval $[t_0, t_0 + \tau]$, then domain (5) preserves its diameter unchanged on this interval, i.e., $D(t) = D(t_0)$ $(t \in [t_0, t_0 + \tau])$.

We now compute the derivative of $V(t, x)$ with respect to t relative to Eq. (1)

$$\frac{dV}{dt} = x^* \left(P_0^* A + A P_0 + \frac{dA}{dt} \right) x + S_p + S_h + S_g$$

$$S_p = x^* \, (\Delta P^* A + A \, \Delta P) \, x \quad (\Delta P = P(t) - P_0)$$
$$S_h = x^* A h + h^* A x, \qquad S_g = x^* A g + g^* A x$$

Bearing (4) in mind and allowing for

$$K^{-1} P_0 K = \operatorname{diag} (\lambda_1, \Lambda_{n-1}), \ K^* P_0^* K = \operatorname{diag} (\bar{\lambda}_1, \Lambda_{n-1}^*)$$

where Λ_{n-1} is an ($n - 1$)st-order matrix whose spectrum coincides with that of matrix P_0 without eigenvalue λ_1, after the change of variables

$$x = K_0 y, \quad y' = (y_1 y'_{n-1})$$

we have

$$\frac{dV}{dt} = 2 \operatorname{Re} \lambda_1 |y_1|^2 + y_{n-1}^* \left(\Lambda_{n-1} + \Lambda_{n-1}^* - 2 \frac{d \ln \varepsilon}{dt} E_{n-1} \right) y_{n-1} + S_p + S_h + S_g$$

Here

$$S_p = y^* \, [\Omega \, (K^{-1} \Delta P K)^* \Omega^{-1} + \Omega^{-1} K^{-1} \Delta P K \Omega] y$$
$$S_h = y^* \Omega^{-1} K^* h + h^* K \Omega^{-1} y, \quad S_g = y^* \Omega^{-1} K^* g + g^* K \Omega^{-1} y$$

Let $\lambda_{\max}$ be the largest eigenvalue of the Hermitian matrix $1/2 \, (\Lambda_{n-1} + \Lambda_{n-1}^*)$. Then

$$\frac{dV}{dt} \leqslant 2 \operatorname{Re} \lambda_1 |y_1|^2 + 2 \left(\lambda_{\max} - \frac{d \ln \varepsilon}{dt} \right) \| y_{n-1} \|^2 + S_p + S_h + S_g$$

We impose the conditions

$$\lambda_{\max} - d \ln \varepsilon \, (t)/dt = -\alpha, \quad \varepsilon \, (t) \leqslant a < 1, \quad t \in [t_0, \ T > t_0]$$

on function $\varepsilon (t)$. These conditions are satisfied, for example, by the function

$$\varepsilon \, (t) = a \exp \, [- \, (\lambda_{\max} + \alpha \, (T - t)] \quad (t \in [t_0, \ T])$$

With such a choice of $\varepsilon (t)$

$$dV/dt \leqslant -2\alpha \, \| y \|^2 + S_p + S_h + S_g$$

Since $S_p |_{t=t_0} = 0$, by continuity we have

$$S_p |_{V=c^2} \leqslant 1/3 \, \alpha c^2$$

for any sufficiently small c, within some finite interval $[t_0, t_0 + \tau] \subset [t_0, T]$. For any sufficiently small number c the inequality

$$S_h |_{V=c^2} \leqslant 1/3 \, \alpha c^2, \quad t \in [t_0, t_0 + \tau]$$

is fulfilled because $h \, (t, x) = 0 \, (x)$. Finally, for any c we can find a sufficiently small number $\eta (c)$ such that

$$S_g |_{V=c^2} \leqslant 1/3 \, \alpha c^2$$

under the condition $| g_i \, (t, x) | \leqslant \eta \, (c)$. By virtue of the above

$$dV/dt |_{V=c^2} \leqslant -\alpha c^2 < 0$$

for any sufficiently small c and $\eta (c)$, and, hence, on the interval $[t_0, t_0 + \tau]$ all the integral curves of system (1), intersecting the surface $V (t, x) = c^2$, intersect it from the outside in. Thus, all the conditions of the stability definitions have been met. The theorem is proved.

Practically all real systems satisfy the hypotheses of the theorem proved, and, therefore, all of them (with infrequent exception) are stable in the sense of Lebedev's definition.

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Translated by N. H. C.